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Shape Control of Beams by Piezoelectric Actuators

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The shape control of beams by piezoelectric actuators is addressed analytically. Solutions are presented for a beam subjected to different boundary conditions. The solutions show how and how much the piezoelectric actuators can influence the shape of a beam. Several case studies are also presented to show the applications of the analytical solutions in the various analyses relevant to shape control of beams by piezoelectric actuators. The limitation of the actuation forces produced by piezoelectric actuators makes it difficult to realize global and local precise shape control.

I. Introduction

SHAPE control of flexible structures by piezoelectric actuators is one of the interesting applications of piezoelectric materials.[1,2] In this application, the actuation forces produced by piezoelectric actuators are used to change the shapes of flexible structures. Some work has been reported on the shape control of beams, plates, and shells during the past few years. Some numerical methods, including the Ritz method,[3,4] finite difference method,[5] and finite element method[6–10] have been used to model and analyze the problems relevant to shape control of flexible structures by piezoelectric actuators. However, due to the approximation of the numerical methods, the reported results can only give rough information of the global shape changes of the structures induced by piezoelectric actuators. They cannot show the detailed local shape change information, such as information on the slope or curvature change of a structure at one point. The detailed local shape change information induced by piezoelectric actuators, which shows how the piezoelectric actuators can change the shape of a structure or what types of shape change the piezoelectric actuators can make to a structure, is important for the design and analysis of such a piezoelectric smart structure. To obtain this information, one needs to analytically solve the problem of shape control. In general, it is very difficult and complicated to solve analytically the problem for a two-dimensional plate; however, a one-dimensional beam is another case, and an analytical solution can be possibly obtained with acceptable complexity. The conclusions for a two-dimensional plate can be drawn from those for a one-dimensional beam.

This paper will present the analytical solutions of the deflection of a beam induced by both piezoelectric actuators and external forces. In accordance with the solutions, the detailed local shape change information induced by piezoelectric actuators will be addressed. Using the solutions, one can analytically perform various analyses relevant to the shape control of beams by piezoelectric actuators. Case studies will also be carried out to show the applications of the solution.

II. Solution

Figure 1 shows a beam with n pairs of piezoelectric actuators bonded on it. Each pair of piezoelectric actuators consists of one actuator bonded on the top surface of the beam and the same type of actuator symmetrically bonded on the bottom surface of the beam. This symmetric distribution of piezoelectric actuators was commonly adopted in previous reports. The length, width, and thickness of the beam are l_0, b, and t_s, respectively. The width and thickness of each actuator are b and t_a, respectively. The left and right end of the kth pair of piezoelectric actuators are l_{k1} and l_{k2}, far away from the left end of the beam. The voltage applied to each actuator of the kth pair of piezoelectric actuators is V_k. The two actuators of the kth pair of piezoelectric actuators are polarized so that a pure bending actuation effect can be produced by V_k, where $k = 1, 2, \ldots, n$. Young's modulus of the beam and the piezoelectric actuators are Y_s and Y_a, respectively. The piezoelectric strain constant of the piezoelectric actuators is d_{31}. N_L and M_L and N_R and M_R are the constrained forces and moments acting on the left and right ends of the beam, respectively. In the analysis of the deflection of the beam given hereafter, plane hypothesis[11] will be adopted. The error induced by this can usually be neglected[12] relative to the other errors arising from nonlinearity (such as hysteresis) and the tolerance of piezoelectric constants,[12] etc. The hysteresis of a PZT actuator is typically on the order of 10–15% of the commanded deformation,[13] and the standard tolerance of piezoelectric constants is 20% of their typical values.[14]

If the excitation of the electric field in one of the preceding piezoelectric actuators is treated as normal external excitation forces acting on the actuator, and simultaneously the actuator is regarded as a pure elastic body, the excitation forces act on the two ends of the actuator with the intensity of $d_{31} Y_a V_k / t_a$ and are in the axis x. Then the excitation of the voltage applied to kth pair of actuators behaves as two equal and opposite bending moments acting on the two ends of the pair of actuators. In accordance with the sign convention adopted in the mechanics of materials,[11] the bending moments M_{Bk} can be obtained by

$$M_{Bk} = -d_{31} b Y_a (t_a + t_s) V_k \tag{1}$$

Then the force and moment equilibrium conditions of the beam give

$$N_L = -N_R \tag{2}$$

$$M_L = M_R + N_R l_0 \tag{3}$$

If the deflection of the beam is $w(x)$ under the condition of small deflection, we have

$$A(x) w''(x) = M(x) \tag{4}$$

where $w''(x)$ denotes the second-order derivative of $w(x)$, $A(x)$ is the bending stiffness, and $M(x)$ is the bending moment of the cross section at a distance x far away from the clamped end of the beam. In Eq. (4), we have

$$A(x) = \begin{cases} Y_s I_s, & \text{for} \quad l_{(k-1)2} \le x < l_{k1} \text{ and } l_{n2} \le x \le l_0 \\ Y_s I_s + 2 Y_a I_a, & \text{for} \quad l_{k1} \le x < l_{k2} \end{cases} \tag{5}$$

where $l_{02} = 0$ and I_s and I_a are the rotary inertia of the beam and the actuators with respect to the middle plane of the beam, respectively,

$$I_s = b t_s^3 / 12, \qquad I_a = b t_a Y_a \left(3 t_s^2 + 6 t_a t_s + 4 t_a^2\right) / 12 \tag{6}$$

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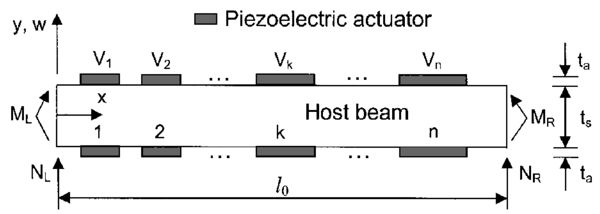

Fig. 1 Beam with *n* pairs of piezoelectric actuators bonded on it.

$$M(x) = \begin{cases} M_R + N_R(l_0 - x) + M_e(x), & \text{for} \quad l_{(k-1)2} \le x < l_{k1} \\ & \text{and} \quad l_{n2} \le x \le l_0 \\ M_{Bk} + M_R + N_R(l_0 - x) + M_e(x), & \text{for} \quad l_{k1} \le x < l_{k2} \end{cases} \tag{7}$$

where $M_e(x)$ is the bending moment independently induced by external actions acting on the beam in addition to the piezoelectric actuation.

Substituting Eqs. (5) and (7) into Eq. (4) and integrating it, we obtain

$$Y_s I_s w'(x) = -N_R x^2/2 + (M_R + N_R l_0)x + G(x) + C_k \qquad \text{for} \quad l_{(k-1)2} \le x \le l_{k1}$$

$$(Y_s I_s + 2Y_a I_a)w'(x) = -N_R x^2/2 + (M_{Bk} + M_R + N_R l_0)x + G(x) + D_k, \quad \text{for} \quad l_{k1} \le x \le l_{k2}$$

$$Y_s I_s w'(x) = -N_R x^2/2 + (M_R + N_R l_0)x + G(x) + C_{n+1} \qquad \text{for} \quad l_{n2} \le x \le l_0 \tag{8}$$

and integrating Eq. (8), we obtain

$$Y_s I_s w(x) = -N_R x^3/6 + (M_R + N_R l_0)x^2/2 + H(x) + C_k x + E_k, \quad \text{for} \quad l_{(k-1)2} \le x \le l_{k1}$$

$$(Y_s I_s + 2Y_a I_a)w(x) = -N_R x^3/6 + (M_{Bk} + M_R + N_R l_0)x^2/2 + H(x) + D_k x + F_k, \quad \text{for} \quad l_{k1} \le x \le l_{k2}$$

$$Y_s I_s w(x) = -N_R x^3/6 + (M_R + N_R l_0)x^2/2 + H(x) + C_{n+1}x + E_{n+1}, \quad \text{for} \quad l_{n2} \le x \le l_0 \tag{9}$$

where $w'(x)$ is the first-order derivative of $w(x)$; C_k, C_{n+1}, D_k, E_k, E_{n+1}, and F_k are the constants of integration and will be determined by the boundary conditions and continuity conditions of the deflection and slope of the beam together with N_R and M_R; and $k = 1, 2, \ldots, n$

$$G(x) = \int M_e(x)\,\mathrm{d}x \tag{10}$$

$$H(x) = \int G(x)\,\mathrm{d}x \tag{11}$$

The results for the constants of integration and N_R and M_R are as the follows. For a beam with clamped–free (CF), clamped–supported (CS), or clamped–clamped (CC) boundary conditions,

$$C_1 = -G(0) \tag{12}$$

$$C_{k+1} = -G(0) + \frac{1}{1+\lambda_1}\sum_{i=1}^{k}\left\{(\lambda_1 M_{Bi} - M_R - N_R l_0)(l_{i2} - l_{i1}) + \frac{N_R\left(l_{i2}^2 - l_{i1}^2\right)}{2} - [G(l_{i2}) - G(l_{i1})]\right\} \tag{13}$$

$$D_k = -\left(1 + \frac{1}{\lambda_1}\right)G(0) - M_{Bk}l_{k2} + \frac{1}{\lambda_1}\left\{\left[M_R + N_R\left(l_0 - \frac{l_{k2}}{2}\right)\right]l_{k2} + G(l_{k2})\right\} + \frac{1}{\lambda_1}\sum_{i=1}^{k}\left\{(\lambda_1 M_{Bi} - M_R - N_R l_0)(l_{i2} - l_{i1}) + \frac{N_R\left(l_{i2}^2 - l_{i1}^2\right)}{2} - [G(l_{i2}) - G(l_{i1})]\right\} \tag{14}$$

$$E_1 = -H(0) \tag{15}$$

$$E_{k+1} = -H(0) + \frac{1}{1+\lambda_1}\sum_{i=1}^{k}\left\{(-\lambda_1 M_{Bi} + M_R + N_R l_0)\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - \frac{N_R\left(l_{i2}^3 - l_{i1}^3\right)}{3} + [G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] - [H(l_{i2}) - H(l_{i1})]\right\} \tag{16}$$

$$F_k = -\left(1 + \frac{1}{\lambda_1}\right)H(0) + \frac{M_{Bk}l_{k2}^2}{2} - \frac{1}{\lambda_1}\left\{\left[\frac{M_R}{2} + N_R\left(\frac{l_0}{2} - \frac{l_{k2}}{3}\right)\right]l_{k2}^2 + G(l_{k2})l_{k2} - H(l_{k2})\right\} + \frac{1}{\lambda_1}\sum_{i=1}^{k}\left\{(-\lambda_1 M_{Bi} + M_R + N_R l_0)\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - \frac{N_R\left(l_{i2}^3 - l_{i1}^3\right)}{3} + [G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] - [H(l_{i2}) - H(l_{i1})]\right\} \tag{17}$$

where

$$\lambda_1 = Y_s I_s / 2Y_a I_a \tag{18}$$

In Eqs. (12–17), $N_R = 0$ and $M_R = 0$ for the CF boundary condition

$$N_R = \frac{p_{11}p_{23} - p_{13}p_{21}}{p_{11}p_{22} - p_{12}p_{21}} \tag{19}$$

$$M_R = \frac{p_{13}p_{22} - p_{12}p_{23}}{p_{11}p_{22} - p_{12}p_{21}} \tag{20}$$

with

$$p_{11} = l_0 - \frac{1}{1+\lambda_1}\sum_{i=1}^{n}(l_{i2} - l_{i1})$$

$$p_{12} = \frac{l_0^2}{2} - \frac{1}{1+\lambda_1}\sum_{i=1}^{n}(l_{i2} - l_{i1})\left(l_0 - \frac{l_{i1} + l_{i2}}{2}\right)$$

$$p_{13} = -\frac{\lambda_1}{1+\lambda_1}\sum_{i=1}^{n}M_{Bi}(l_{i2} - l_{i1}) + \frac{1}{1+\lambda_1}\sum_{i=1}^{n}[G(l_{i2}) - G(l_{i1})] - [G(l_0) - G(0)]$$

$$p_{21} = \frac{l_0^2}{2} - \frac{1}{1+\lambda_1}\sum_{i=1}^{n}\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2}$$

$$p_{22} = \frac{l_0^3}{6} - \frac{1}{1+\lambda_1}\sum_{i=1}^{n}\left(\frac{l_{i2}^2 - l_{i1}^2}{2}l_0 - \frac{l_{i2}^3 - l_{i1}^3}{3}\right)$$

$$P_{23} = -\frac{\lambda_1}{1+\lambda_1}\sum_{i=1}^{n} M_{Bi}\frac{l_{i2}^2 - l_{i1}^2}{2} + \frac{1}{1+\lambda_1}\sum_{i=1}^{n}\{[G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] - [H(l_{i2}) - H(l_{i1})]\} - G(l_0)l_0 + [H(l_0) - H(0)] \tag{21}$$

for the CC boundary condition, and $M_R = 0$,

$$N_R = p_{32}/p_{31} \tag{22}$$

with

$$p_{31} = \frac{l_0^3}{3} + \frac{1}{1+\lambda_1}\sum_{i=1}^{n}\left[\left(l_{i2}^2 - l_{i1}^2\right)l_0 - \frac{l_{i2}^3 - l_{i1}^3}{3} - l_0^2\right]$$

$$p_{32} = \frac{\lambda_1}{1+\lambda_1}\sum_{i=1}^{n} M_{Bi}(l_{i2} - l_{i1})\left(\frac{l_{i1}+l_{i2}}{2} - l_0\right) + \frac{1}{1+\lambda_1}\sum_{i=1}^{n}\{[G(l_{i2}) - G(l_{i1})]l_0 - [G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] + [H(l_{i2}) - H(l_{i1})]\} + G(l_0)l_0 - [H(l_0) - H(0)] \tag{23}$$

for the CS boundary condition.

For a beam with a simply supported (SS) boundary condition, $N_R = 0$ and $M_R = 0$, and

$$C_k = -\frac{1}{l_0}[H(l_0) - H(0)] + \frac{1}{(1+\lambda_1)l_0}\left\{\sum_{i=1}^{n}\left[\lambda_1 M_{Bi}\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - (G(l_{i2})l_{i2} - G(l_{i1})l_{i1}) + [H(l_{i2}) - H(l_{i1})]\right] - \sum_{i=k}^{n} l_0\{\lambda_1 M_{Bi}(l_{i2} - l_{i1}) - [G(l_{i2}) - G(l_{i1})]\}\right\} \tag{24}$$

$$C_{n+1} = -\frac{1}{l_0}[H(l_0) - H(0)] + \frac{1}{(1+\lambda_1)l_0}\sum_{i=1}^{n}\left\{\lambda_1 M_{Bi}\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - [G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] + [H(l_{i2}) - H(l_{i1})]\right\} \tag{25}$$

$$D_k = -\frac{1}{l_0}\left(1 + \frac{1}{\lambda_1}\right)[H(l_0) - H(0)] - M_{Bk}l_{k1} + \frac{1}{\lambda_1}G(l_{k1}) + \frac{1}{l_0}\sum_{i=1}^{n}\left\{M_{Bi}\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - \frac{1}{\lambda_1}\{(G(l_{i2})l_{i2} - G(l_{i1})l_{i1}) - [H(l_{i2}) - H(l_{i1})]\}\right\} - \sum_{i=k}^{n}\left\{M_{Bi}(l_{i2} - l_{i1}) - \frac{1}{\lambda_1}[G(l_{i2}) - G(l_{i1})]\right\} \tag{26}$$

$$E_1 = -H(0) \tag{27}$$

$$E_{k+1} = -H(0) - \frac{1}{1+\lambda_1}\sum_{i=1}^{k}\left\{\lambda_1 M_{Bi}\frac{\left(l_{i2}^2 - l_{i1}^2\right)}{2} - [G(l_{i2})l_{i2} - G(l_{i1})l_{i1}] + [H(l_{i2}) - H(l_{i1})]\right\} \tag{28}$$

$$F_k = -\left(1 + \frac{1}{\lambda_1}\right)H(0) - \frac{1}{\lambda_1}[G(l_{k2})l_{k2} - H(l_{k2})] + \frac{M_{Bk}l_{k2}^2}{2} - \sum_{i=1}^{k}\left\{M_{Bi}\frac{l_{i2}^2 - l_{i1}^2}{2} - \frac{1}{\lambda_1}\{(G(l_{i2})l_{i2} - G(l_{i1})l_{i1}) - [H(l_{i2}) - H(l_{i1})]\}\right\} \tag{29}$$

Thus far, the solutions of the deflection of the beam under four boundary conditions have been given. Using these solutions, one can analytically analyze the static deformation of a beam subject to both piezoelectric actuation and other mechanical action. In accordance with the solutions, one can also choose the physical parameters, voltages, sizes, positions, and number of the actuators to satisfy one's needs, that is, to control the shape of a beam. Hence, various control problems, such as the deflection or slope control of some points, the constrained forces and moments control applied by the boundary, precision local shape control around a point, desired whole shape control of a beam, etc., on the shape control or optimal shape control of a beam can be solved analytically. In the next section of this paper, several examples are given to show the applications of the solutions.

By dropping the terms including G and H in the preceding equations, we can get the solution of the deflection of a beam independently induced by piezoelectric actuators. Of course, one can also obtain the deflection of the beam by superposition of the two deflections independently induced by the piezoelectric actuators and the other actions, respectively. From Eq. (9), one can see that the piezoelectric actuators can only deform a beam by a quadratic curve for the cases of the CF and SS boundary conditions or by a cubic curve for the cases of the CC and CS boundary conditions. That is, the piezoelectric actuators cannot arbitrary change the shape of a beam or cannot deform a beam to arbitrary shapes. This reveals the limitation of shape control by piezoelectric actuators. To achieve arbitrary shape control of a beam, only optimal design on the piezoelectric actuators can be conducted to minimize the difference between the actually achieved shape and the desired shape.

For the convenience of later applications, here we specifically write out the solutions of the deflection of a beam with one pair of piezoelectric actuators bonded on it (shown in Fig. 2). For the CF boundary condition, we have

$$w(x) = \begin{cases} 0, & \text{for } 0 \le x \le l_1 \\ (M_B/2\lambda_2)(x - l_1)^2, & \text{for } l_1 \le x \le l_2 \\ (M_B/\lambda_2)(l_2 - l_1)[x - (l_1 + l_2)/2] & \text{for } l_2 \le x \le l_0 \end{cases} \tag{30}$$

where

$$\lambda_2 = Y_s I_s + 2Y_a I_a \tag{31}$$

$$M_B = -d_{31}bY_a(t_a + t_s)V \tag{32}$$

with V the voltages applied to each actuator.

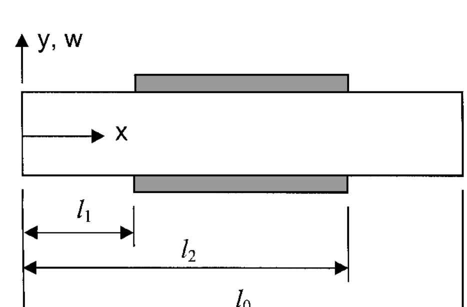

Fig. 2 Beam with one pair of piezoelectric actuators bonded on it.

For the SS boundary condition, we have

$$
w(x) = \begin{cases} \dfrac{M_B}{\lambda_2}(l_2 - l_1)\left(\dfrac{l_1 + l_2}{2l_0} - 1\right)x, & \text{for} \quad 0 \le x \le l_1 \\ \dfrac{M_B}{2\lambda_2}\left[x^2 + \left(\dfrac{l_2^2 - l_1^2}{l_0} - 2l_2\right)x + l_1^2\right], & \text{for} \quad l_1 \le x \le l_2 \\ \dfrac{M_B}{2\lambda_2}\left(l_2^2 - l_1^2\right)\left(\dfrac{x}{l_0} - 1\right), & \text{for} \quad l_2 \le x \le l_0 \end{cases} \tag{33}
$$

For the CC boundary condition, we have

$$
w(x) = \frac{1}{Y_s I_s}\left[-\frac{N_R x^3}{6} + \frac{(M_R + N_R l_0)x^2}{2}\right] \quad \text{for} \quad 0 \le x \le l_1
$$

$$
\begin{aligned}
w(x) = \frac{1}{\lambda_2}\Bigg\{ & -\frac{N_R x^3}{6} + \frac{(M_B + M_R + N_R l_0)x^2}{2} \\
& + \left\{-M_B + \frac{1}{\lambda_1}\left[M_R + N_R\left(l_0 - \frac{l_1}{2}\right)\right]\right\} l_1 x \\
& + \left\{M_B - \frac{1}{\lambda_1}\left[M_R + N_R\left(l_0 - \frac{2l_1}{3}\right)\right]\right\}\frac{l_1^2}{2}\Bigg\}
\end{aligned}
$$

$$
\text{for} \quad l_1 \le x \le l_2
$$

$$
\begin{aligned}
w(x) = \frac{1}{Y_s I_s}\Bigg\{ & -\frac{N_R x^3}{6} + \frac{(M_R + N_R l_0)x^2}{2} \\
& + \frac{l_2 - l_1}{1 + \lambda_1}\left[\lambda_1 M_B - M_R - N_R\left(l_0 + \frac{l_1 + l_2}{2}\right)\right]x \\
& - \frac{1}{2(1 + \lambda_1)}\Bigg[(\lambda_1 M_B - M_R - N_R l_0)\left(l_2^2 - l_1^2\right) \\
& + \frac{2N_R\left(l_2^3 - l_1^3\right)}{3}\Bigg]\Bigg\}, \quad \text{for} \quad l_2 \le x \le l_0
\end{aligned} \tag{34}
$$

where

$$
N_R = \frac{p_{11}^{(1)} p_{23}^{(1)} - p_{13}^{(1)} p_{21}^{(1)}}{p_{11}^{(1)} p_{22}^{(1)} - p_{12}^{(1)} p_{21}^{(1)}} \tag{35}
$$

$$
M_R = \frac{p_{13}^{(1)} p_{22}^{(1)} - p_{12}^{(1)} p_{23}^{(1)}}{p_{11}^{(1)} p_{22}^{(1)} - p_{12}^{(1)} p_{21}^{(1)}} \tag{36}
$$

with

$$
p_{11}^{(1)} = l_0 - \frac{1}{1 + \lambda_1}(l_2 - l_1)
$$

$$
p_{12}^{(1)} = \frac{l_0^2}{2} - \frac{1}{1 + \lambda_1}(l_2 - l_1)\left(l_0 - \frac{l_1 + l_2}{2}\right)
$$

$$
p_{13}^{(1)} = -\frac{\lambda_1}{1 + \lambda_1}M_B(l_2 - l_1), \qquad p_{21}^{(1)} = \frac{l_0^2}{2} - \frac{1}{1 + \lambda_1}\frac{\left(l_2^2 - l_1^2\right)}{2}
$$

$$
p_{22}^{(1)} = \frac{l_0^3}{6} - \frac{1}{1 + \lambda_1}\left(\frac{l_2^2 - l_1^2}{2}l_0 - \frac{l_2^3 - l_1^3}{3}\right)
$$

$$
p_{23}^{(1)} = -\frac{\lambda_1}{1 + \lambda_1}M_B\frac{l_2^2 - l_1^2}{2} \tag{37}
$$

For the CS boundary condition, the expression of $w(x)$ is the same as Eq. (34), except that $M_R = 0$ and

$$
N_R = \frac{p_{32}^{(1)}}{p_{31}^{(1)}} \tag{38}
$$

with

$$
p_{31}^{(1)} = \frac{l_0^3}{3} + \frac{1}{1 + \lambda_1}\left[\left(l_2^2 - l_1^2\right)l_0 - \frac{l_2^3 - l_1^3}{3} - l_0^2\right]
$$

$$
p_{32}^{(1)} = \frac{\lambda_1}{1 + \lambda_1}M_B(l_2 - l_1)\left(\frac{l_1 + l_2}{2} - l_0\right) \tag{39}
$$

III. Cases Studies and Discussion

In accordance with the solution shown in Eq. (30), one can directly write the tip deflection of a piezoelectric bimorph beam as $3d_{31}l_0V/(4t_a^2)$, as is widely known.[1] The deflection of the [45/ −45/ −45/45] composite cantilever beam subjected to an actuator voltage of 75 V, which has been calculated by Donthireddy and Chandrashekhara[7] with their finite element model, obtained with Eq. (30), is shown in Fig. 3. The results obtained here show good agreement with those of Donthireddy and Chandrashekhara.[7] The difference between their results and the results here may stem from their having considered the lateral strains in their one-dimensional beam model.

Figure 4 shows the normalized deflections $[w(x)/w_0$, where $w_0 = M_B l_0^2/\lambda_2]$ of a beam for the four different boundary conditions,

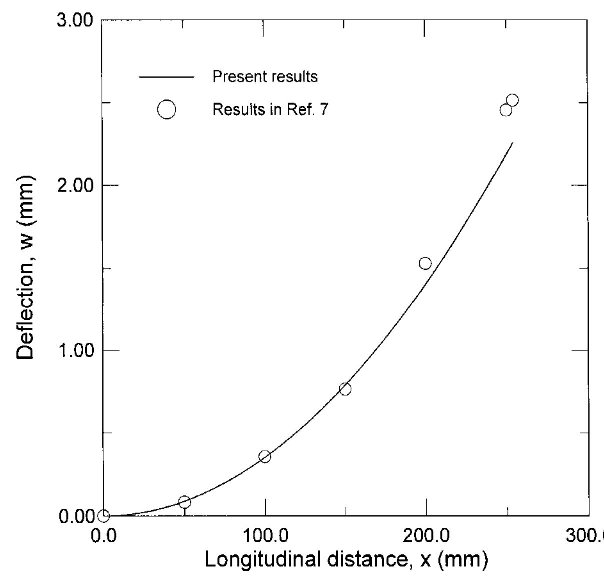

Fig. 3 Deflection of a [45/− 45/− 45/45] cantilever beam for an actuator voltage of 75 V.

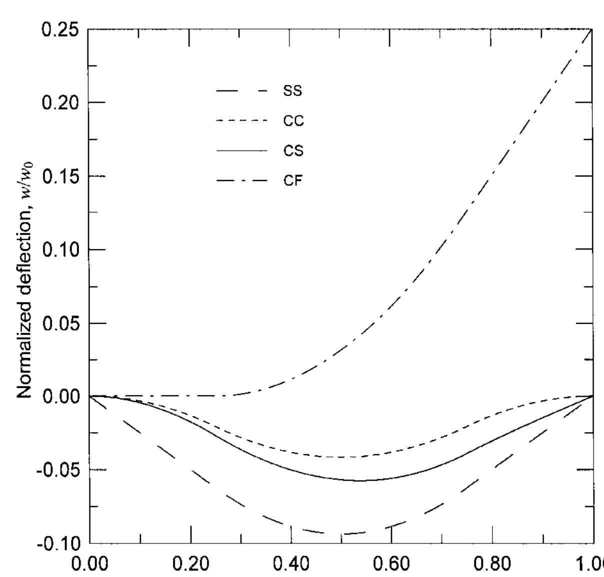

Fig. 4 Deflections of the beam shown in Fig. 2 for different boundary conditions.

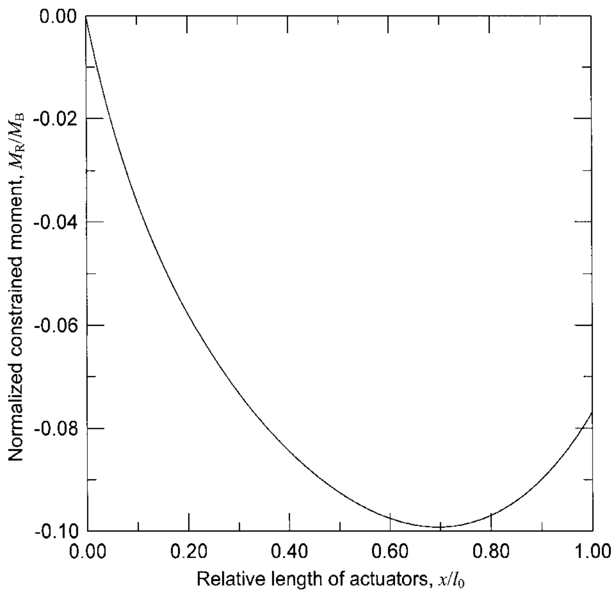

Fig. 5 Dependence of the constrained moment of a CC beam on the length of the actuators.

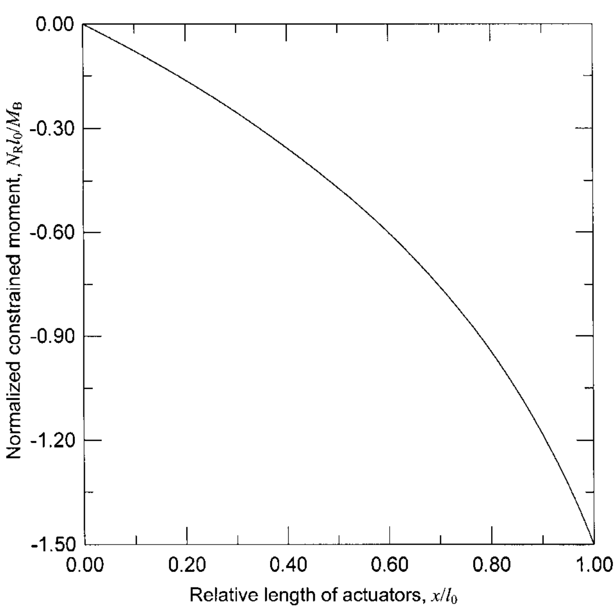

Fig. 6 Dependence of the constrained force of a CS beam on the length of the actuators.

respectively. The beam is actuated independently by one pair of piezoelectric actuators with parameters (see Fig. 2), $\lambda_1 = 1$, $l_1 = l_0/4$, and $l_2 = 3l_0/4$. In Fig. 4, the longitudinal axis is the relative longitudinal distance x/l_0. Figure 4 shows clearly how one pair of actuators deform a beam and what differences arise between the deformations for different boundary conditions.

We consider another beam actuated by one pair of actuators whose center is the same as that of the beam, that is, $l_1 + l_2 = l_0$. Figure 5 shows the dependence of the normalized constrained moment M_R/M_B acting on the end of such a beam with a CC boundary condition on the relative length, $(l_2 - l_1)/l_0$, of the actuators. Note that the constrained force is zero for this CC beam. From Fig. 5, we can see that a maximum of the constrained moment occurs at about an actuator length $0.7l_0$. Thus, if one wants to design such a structure to actuate its base, one should adopt this actuator length to maximize the actuation efficiency. Figure 6 shows the dependence of the normalized constrained force $N_R l_0/M_B$ acting on the right end of a beam with a CS boundary condition on the relative length of the actuators. From Fig. 6, we can see that the constrained force increases with increasing actuator length.

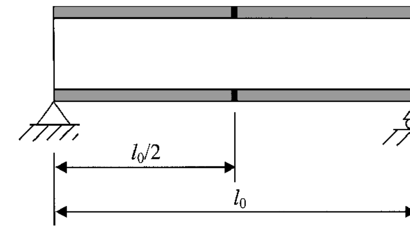

Fig. 7 SS beam covered by two pairs of actuators.

We consider a CC beam actuated by one pair of actuators whose center is same as that of the beam. A force P in the direction $-y$ acts on the middle point of the beam. Now we try to find the actuator length and actuator voltage to control the middle point deflection and constrained moment of the beam to zero. This is another type of control application that can be realized by piezoelectric actuators. The constrained moment and deflection of the beam independently induced by P, represented by M_{R2} and $w_2(x)$, respectively, can be obtained as

$$M_{R2} = -\alpha P \tag{40}$$

$$w_2(x) = \begin{cases} (P/12Y_s I_s)(x - 6\alpha)x^2, & \text{for } 0 \le x \le l_1 \\ (P/12\lambda_2)\big[x^3 - 6x^2 + (3/\lambda_1)(l_1 - 4\alpha)l_1 x \\ \quad + (2/\lambda_1)(3\alpha - l_1)l_1^2\big], & \text{for } l_1 \le x \le l_0/2 \end{cases} \tag{41}$$

where

$$\alpha = \frac{4l_1^2 + \lambda_1 l_0^2}{8(2l_1 + \lambda_1 l_0)} P \tag{42}$$

Because of symmetry, Eq. (41) only gives the expressions for $0 \le x \le l_0/2$. The constrained moment and deflection of the beam independently induced by the actuators, represented by M_{R1} and $w_1(x)$, respectively, can be obtained from Eq. (34). Then the problem can be expressed by the two equations

$$M_{R1} + M_{R2} = 0 \tag{43}$$

$$w_1(l_0/2) + w_2(l_0/2) = 0 \tag{44}$$

Solving Eqs. (43) and (44), one can obtain the actuator length and the actuation bending moment. For example, one can get $l_2 - l_1 = 0.559l_0$ and $M_B = -1.08l_0 P$ for the case in which $\lambda_1 = 1$.

Next, we consider an SS beam (shown in Fig. 7). We hope to deform the beam to a desired shape

$$y_1(x) = y_{10} \sin(2\pi/l_0)x \tag{45}$$

In accordance with the preceding analysis and solution, we adopt the distribution of piezoelectric actuators shown in Fig. 7, that is, the beam is wholly covered by two pairs of actuators of equal size. Based on Eqs. (9) and (24–29), the deflection induced by the actuators for this case can be written as

$$w(x) = \begin{cases} \dfrac{1}{\lambda_2}\left[\dfrac{M_{B1}}{2}x^2 - \dfrac{3M_{B1} + M_{B2}}{8}l_0 x\right], & \text{for } 0 \le x \le \dfrac{l_0}{2} \\ \dfrac{1}{\lambda_2}\left[\dfrac{M_{B2}}{2}x^2 + \dfrac{M_{B1} - 5M_{B2}}{8}l_0 x - \dfrac{M_{B1} - M_{B2}}{8}l_0^2\right] \\ \qquad \text{for } \dfrac{l_0}{2} \le x \le l_0 \end{cases} \tag{46}$$

The error between the achieved shape and desired shape is quantified by

$$J = \int_0^{l_0} [w(x) - y_1(x)]^2 \, dx \tag{47}$$

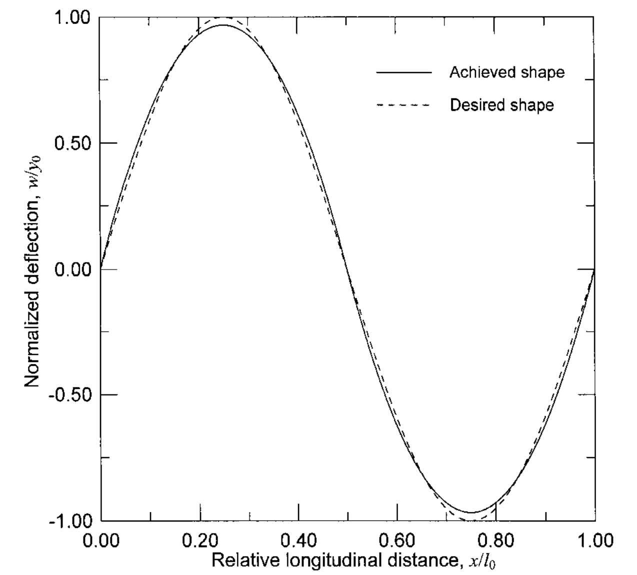

Fig. 8 Optimal achieved and desired shapes of the beam shown in Fig. 7.

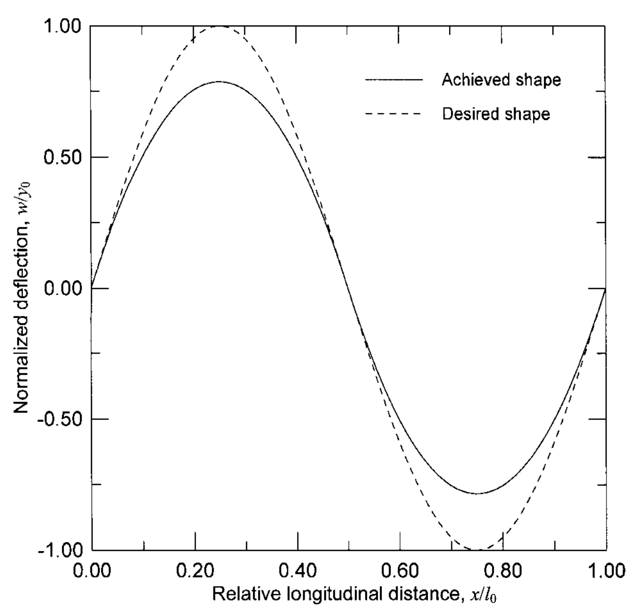

Fig. 9 Achieved and desired shapes of the beam shown in Fig. 7 subjected to some constraints on its middle point.

To optimally control the shape of the beam, that is, to minimize the error, the following conditions should be satisfied:

$$\frac{\partial J}{\partial M_{B1}} = \frac{\partial J}{\partial M_{B2}} = 0 \tag{48}$$

From Eq. (48), we get

$$M_{B2} = -M_{B1} = \frac{960\lambda_2 y_{01}}{\pi^3 l_0^2} \tag{49}$$

Figure 8 shows the normalized achieved and desired shapes $[w(x)/y_{01}]$ of the beam for this situation. For a set of parameters $l_0 = 200$ mm, $b = 15$ mm, $t_a = 0.2$ mm, $t_s = 1$ mm, $Y_a = 63$ GPa, $Y_s = 70$ GPa, $d_{31} = -254 \times 10^{-12}$ m/V, and $y_{01} = 0.2$ mm, we obtain the voltages needed to be applied to the actuators as $V_2 = -V_1 = 120.84$ V. If one wants to deform the beam so that the deflection and slope at the point $x = l_0/2$ equals those of $y_1(x)$, one can also get

$$M_{B2} = -M_{B1} = 8\pi\lambda_2 y_{01}\big/l_0^2 \tag{50}$$

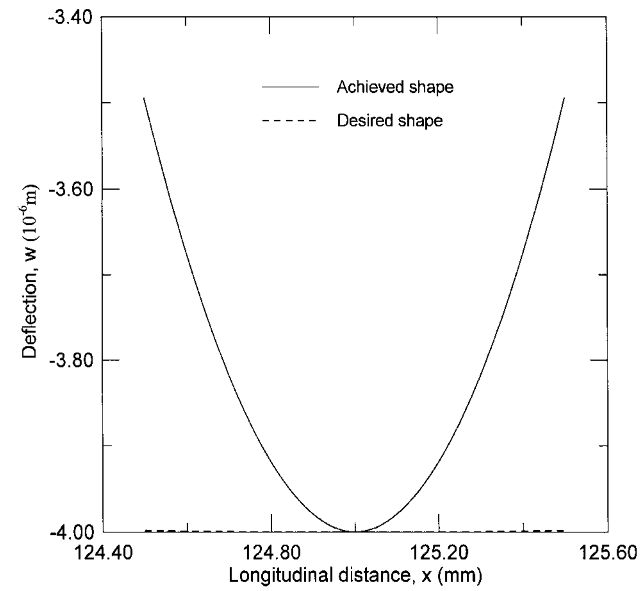

Fig. 10 Local achieved and desired shapes of a CS beam with four pairs of actuators.

and $V_2 = -V_1 = 98.09$ V. Figure 9 shows the normalized achieved and desired shapes of the beam for this situation.

Finally, we investigate a local shape control problem. For a CS beam, we try to make the shape of the beam at $x = x_0 = 5l_0/8$ approach that of

$$y_2(x) = -y_{02}\sin(4\pi/l_0) \tag{51}$$

as nearly as possible. In accordance with the preceding solution, using the piezoelectric actuators one can only realize the third approach to an arbitrary desired shape of a beam. Thus, we use four pairs of actuators to control the shape of the beam. The four pairs of actuators are the same and wholly cover the beam. The deflection of the beam $l_0/2 \le x \le 3l_0/4$ can be concretely written as

$$w(x) = \frac{1}{64\lambda_2}\Bigg[\left(-\frac{x^3}{l_0} + 3x^2\right)\sum_{i=1}^{4}(2i-9)M_{Bi} + 32M_{B3}x^2 + 16(M_{B1} + M_{B2} - 2M_{B3})l_0 x + 2(-M_{B1} - 3M_{B2} + 4M_{B3})l_0^2\Bigg] \tag{52}$$

The following equations should be satisfied to realize the third approach:

$$w(x_0) = y_2(x_0) \tag{53}$$

$$w'(x_0) = y_2'(x_0) \tag{54}$$

$$w''(x_0) = y_2''(x_0) \tag{55}$$

$$w'''(x_0) = y_2'''(x_0) \tag{56}$$

By solving the group of Eqs. (53–56), one can get the voltages to be applied to the actuators. For the preceding set of beam parameters and $y_{02} = 4$ μm, we get $V_1 = 1182.1$, $V_2 = -2759.9$, $V_3 = 3155.6$, and $V_4 = -3942.0$ V. Theses voltages are too large for real applications in many respects. Thus, it is difficult to realize the local third approach of transforming a beam shape to an arbitrary desired shape. Even though one can realize the local third approach at one point, one cannot ensure that the difference between the achieved and the desired shapes near the point is small enough. The local significant difference (shown in Fig. 10) near the point $x = x_0 = 5l_0/8$ between the achieved and desired shapes of the CS beam calculated with the actuator voltages obtained earlier illustrates this. The difference between the achieved and desired shapes shown in the results of

Varadarajan et al. (see Fig. 5 in Ref. 10) is also significant. Hence, one needs to choose the number, positions, and sizes of the actuators wisely, and some optimal design algorithm should be used to design the actuators.

IV. Conclusions

The solutions of the deflection of a beam simultaneously induced by piezoelectric actuators and other external actions have been analytically given for different boundary conditions. The solutions for a beam with other structures can be similarly given, and the results will also be similar. The solutions show that the piezoelectric actuators can only deform a beam by a quadratic or cubic curve due to their actuation bending moments occurring at the ends of the actuators in a pair form. Because of the same characteristics of piezoactuation, this conclusion can be extended to the shape control of plates by piezoelectric actuators. That is, the piezoelectric actuators can only deform a plate by a quadratic or cubic surface. Several cases were studied to show the applications of the solutions to analyzing various shape control problems. Numerical results show that it is difficult to make the shape of a beam locally approach a desired shape with piezoelectric actuators.

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